We investigate here the statically indeterminate or “muscle force distribution” problem, using the elbow as an example. The question is: how do our calculated values of joint contact force depend on the type of model assumed, and in particular, on the number of muscles assumed to act? Figure 1 shows the elbow in 90° flexion supporting a weight $W=25$ N, with three muscles acting. The weight of the forearm is 17 N, and acts at 13 cm from the elbow joint. Assume also that the moment arms (perpendicular distance from line of muscle action to joint center) are 4.3, 3.2, and 7.2 cm about the elbow joint for the biceps brachii, brachialis, and brachioradialis, respectively, and that the muscle angles $\theta$ are as shown in Table 1.

A. Assuming that the physiological cross-sectional areas (PCSA) are as shown in Table 1, and using the arguments of muscle scaling, compare the values of the joint contact force calculated with all muscles acting to that value obtained by assuming just the action of one muscle. Do this separately for each of the three muscles.

Table 1: Physiological cross-sectional areas for elbow joint flexors (Adapted from *Basic Orthopaedic Biomechanics* by Mow and Hayes (editors), page 33).

<table>
<thead>
<tr>
<th>Muscle</th>
<th>Angle $\theta$ (degrees)</th>
<th>PCSA (cm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biceps brachii (BIC)</td>
<td>80</td>
<td>4.5</td>
</tr>
<tr>
<td>Brachialis (BRA)</td>
<td>69</td>
<td>7.0</td>
</tr>
<tr>
<td>Brachioradialis (BRR)</td>
<td>23</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Figure 1: Elbow flexion problem, which is statically indeterminate.

B. A second approach to this problem is to use numerical optimization techniques. We will use MATLAB for this. The task here is to choose various forms of the optimization criterion and compare the resulting predictions of the joint contact force. We will formulate different types of linear and non-linear optimization criteria since linear and
non-linear functions have different mathematical characteristics in optimization problems. These “cost” or “objective” functions should be functions of the three muscle forces, $F_{\text{BRA}}$, $F_{\text{BRC}}$, and $F_{\text{BRR}}$, for which your MATLAB code will compute the magnitude. Once the values of these muscle forces are known, you will compute the corresponding joint contact force and compare it for the different assumed optimization criteria.

Conceptually, this is like an engineering design problem, where the muscle forces can be thought of the design variables. We will use optimization routines to find the design variables that minimize our assumed objective function. For example, we might want to find the diameter and material properties of a structure — the design variables — that minimize its weight — the objective function. Another attribute of our problem is its constrained nature, i.e. the solution must satisfy some other condition beyond minimizing the objective function. For example, when we find the diameter and material properties of a structure that minimize its weight, there may be a limit on the minimum diameter so the structure doesn’t break. This limit is treated mathematically as a design constraint. We call this a “constrained optimization” problem.

Most interesting optimization problems have a number of design variables and need to satisfy a number of constraints. In our case, we have three muscle forces (design variables) and many constraints. For example, we treat the static equilibrium equations as constraints since they specify that some mathematical function of the muscle (and other) forces must equal zero. And since muscles can only apply tension, all muscles forces are constrained to be greater than zero (signifying tensile forces). These are examples of “equality” and “inequality” constraints, respectively.

C. Write a short report (two pages text maximum, double spaced; figures and references extra) that contains:

i) A statement of the overall, collective goal of parts A and B

ii) A paragraph explaining the biomechanical/physiological rationale for each objection function that you used in part B

iii) A table containing the magnitudes of the joint contact and muscle forces resulting from the four optimizations

iv) A brief discussion of the results in the optimization analysis. For example, based on the results, which objective function seems most reasonable and why?

v) A brief discussion of which approach to this problem (muscle scaling or optimization) is most reasonable. Support your analysis with a discussion of the strengths and/or limitations of each approach.
MATLAB TIPS:

i) Formulate four objective functions — you choose them — that are to be minimized. They should be written as functions of the three muscle forces, $F_{BRA}$, $F_{BIC}$, and $F_{BRR}$. For example, if you want to include the joint contact force in your objective function, express it in terms of the three muscle forces via the equations of static equilibrium. Two of the objective functions should be linear (i.e. mathematically linear combinations of the forces), and the other two should be nonlinear (e.g. quadratic, cubic, power law equations, etc.).

ii) Formulate the constraints for your objective functions taking into account the following. Both the muscle forces and the joint contact force are unknown. For this problem, we will calculate the joint contact force directly from the force equilibrium equations once the muscle forces have been determined through optimization. Therefore, the force equilibrium equations will not be used as constraints for the optimization formulation. Instead, force equilibrium will be satisfied by calculating the joint contact force post-hoc. Thus, the moment equilibrium equation will be our only equality constraint (take moments about the joint center for mathematical — and therefore coding — convenience). If you think you have more than one equality constraint, please let us know and let’s discuss it!

Coding constraints in MATLAB: Constraints need to be coded so that they are less than (inequality) or equal to (equality) zero. Thus, if your desired equality constraint is $F_1 + F_2 + F_3 = mg$, code the constraint to be $F_1 + F_2 + F_3 - mg = 0$. Similarly if your desired inequality constraint is $F_1 \geq 0$, code it to be $-F_1 \leq 0$.

Your files: In the course website, find three files: elbow.m, fun.m, and constraints.m.

fun.m is for your objective function and constraints.m is for your constraints. Open the files and add code for your objective functions and constraints in the designated locations. The three muscle forces to be optimized are contained in the vector x. Remember to be consistent about which force is represented by $x(1)$, $x(2)$, and $x(3)$ and comment this clearly in your code (so we can help with debugging, if necessary).

elbow.m contains the function call for the optimization algorithm and most of the coding is done for you. In elbow.m, add code to calculate the magnitude of the joint contact force — this is our eventual output parameter. The optimized muscle forces will all be contained in the vector x after the function call is made.

Once you have edited the files, simply type in elbow at the MATLAB command prompt. This will run the optimization algorithm and return the results. Just edit fun.m for each of your objective functions.