3.6 Extended Surface Heat Transfer

The rate at which heat can be transferred by conduction, convection, or radiation to or from a given body is in direct proportion to the surface area of heat exchange. Thus, surfaces are "extended" in a variety of ways in order to enhance the heat exchange process.

"Finned" surfaces are typical examples of this. An extended surface is often times termed "a fin."

In our past work, we've considered heat from the boundaries of a solid to be in the same direction as the heat by conduction in the solid.

In contrast, for extended surfaces, the direction of heat transfer from the boundaries is orthogonal to the principal direction of hot within the solid.
* Straight, annular, pin fin/pipe

For fin analysis, we assume:

- HT. 1-D in x direction (Fin is thin!)
- SS
- Neglect radiation
- No heat generation
- h is constant over surface

Energy Balance on slice of fin:

\[ E_{in} - E_{out} + \frac{\partial}{\partial x} \int_{x}^{x+dx} q(x) \, dx = \dot{q}_{gen} \]  \hspace{1cm} (SS)

\[ \dot{E}_{in} = \dot{E}_{out} \]

\[ \dot{q} \, dx = \dot{q}_{x+dx} + d\dot{q}_{conv} \]

- Fourier's law:
  \[ \dot{q} = -k \frac{dT}{dx} \]  \hspace{1cm} (A)

Also:

\[ \text{Other notes...} \]
\[ P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \]

Substituting \( x + dx \):

\[ q_{x+dx} = q_x + \frac{dq}{dx} \text{ (Taylor Series)} \]

Substitute \( A \rightarrow B \):

\[ q_{x+dx} = -kA \frac{dT}{dx} - k \frac{d}{dx} \left( A \frac{dT}{dx} \right) \]

With convection transfer described as:

\[ dq = h dA_s (T-T_\infty) \]

Substitute \( d \rightarrow 0 \):

\[ \frac{d}{dx} \left( A \frac{dT}{dx} \right) - \frac{h}{k} \frac{dA_s}{dx} (T-T_\infty) = 0 \]

Product rule!

Substitute \( C, D, E \) into \( 0 \):

\[ \frac{d^2T}{dx^2} + \left( \frac{1}{Ac} \frac{dA_c}{dx} \right) \frac{dT}{dx} - \left( \frac{h}{Ac} \frac{dA_s}{dx} \right) \]

\[ (T-T_\infty) \]

Using \( E \) above, any for given and properties may be added to Temp dist obtained.
Figure 3.14  Fin configurations. (a) Straight fin of uniform cross section. (b) Straight fin of nonuniform cross section. (c) Annular fin. (d) Pin fin.
Fin Geometries + Lhs. Eqn. Solns

\[ T_{\text{conv}}, h \]

\[ T_b \]

\[ q_{\text{conv}} \]

\[ A_c(x) \]

\[ A_c \]

\[ A_s = P_x = A_s(x) \]

\[ P = \text{Perimeter} = 2w + 2t \]

\[ A_c = \text{Cross-sec area} = wt \]

With each fin attached to the base.

@ a temperature \( T(0) = T_b \) and submerged in a fluid @ \( T = T_{\infty} \).

Using eqn 5 from L8 to describe the temperature distribution.

\[
\frac{d^2 T}{dx^2} + \left( \frac{1}{A_c} \frac{dA_c}{dx} \right) \left( \frac{dT}{dx} \right) - \left( \frac{1}{A_c k} \frac{dA_s}{dx} \right) (T_s - T_{\infty}) = 0
\]

for a fin geometry (w/ const. cross-sec area) as shown above:

1. \( \frac{dA_c}{dx} = 0 \) since \( A_c = \text{const.} \)

2. \( \frac{dA_s}{dx} = \frac{d}{dx} (P_x) = P \)
Our governing eqn. become
\[
\frac{d^2T}{dx^2} - \frac{hP}{kAc} (T - T_\infty) = 0
\]

\[\Theta(x) = T(x) - T_\infty \quad \text{since} \quad \frac{\Theta}{x} = \frac{dT}{dx}\]

Tern:
\[
\frac{d^2\Theta}{dx^2} - m^2 \Theta = 0
\]

If:
\[
m^2 = \frac{hP}{kAc}
\]

Eqn (F) is a linear, homogenous, second order, differential equation.
\[
\frac{d^2\Theta}{dx^2} = m^2 \Theta
\]

Wanting a soln. \( \Theta \) which has its second derivative wrt. \( x \) equal to a constant times itself:

Try:
\[
\Theta = e^{\beta x}
\]

Sub.
\[
\frac{d^2\theta}{dx^2} = m^2 \theta
\]

\[
\frac{d^2}{dx^2}(e^{\beta x}) = m^2 e^{\beta x}
\]

\[
\beta \frac{d}{dx}(e^{\beta x}) = m^2 e^{\beta x}
\]

\[
\beta^2 e^{\beta x} = m^2 e^{\beta x}
\]

\[
\beta^2 = m^2 \quad ; \quad \beta = \pm m
\]

\[
\theta = e^{\pm mx}
\]

Thus, the general solution is a combination of constant multiples of \(e^{mx}\)

\[
\therefore \theta(x) = c_1 e^{mx} + c_2 e^{-mx}
\]  \(H\)

We may check by substitution (H) \(\rightarrow\) (G):

\[
\frac{d^2}{dx^2}(c_1 e^{mx} + c_2 e^{-mx}) = m^2 (c_1 e^{mx} + c_2 e^{-mx})
\]

\[
c_1 m^2 e^{mx} + c_2 m^2 e^{-mx} = m^2 (c_1 e^{mx} + c_2 e^{-mx})
\]

\[
\therefore m^2 (c_1 e^{mx} + c_2 e^{-mx}) = m^2 (c_1 e^{mx} + c_2 e^{-mx})
\]

Now, we need appropriate B.C.'s to evaluate: \(c_1 + c_2\)
One B.C. we will use is that of the fin base temperature:
\[ @x=0 ; \quad \theta(0) = T_b - T_\infty \]
\[ \theta(0) = \theta_b \]

The second condition is that of the fin tip @ \( x=L \). Here we will consider 4 different cases:

**CASE A: CONVECTION AT THE TIP**

\[ @x=L ; \quad hA_c[T(L) - T_\infty] = -kA_c \frac{dT}{dx} \left|_{x=L} \right. \]

\[ \frac{dT}{dx} \left|_{x=L} \right. = h \theta(L) = \left. -k \frac{d\theta}{dx} \right|_{x=L} \]

(eg. convection off the tip = conduction that reaches the tip).

Sub 4 into 1 or 5 B.C.'s

\[ @x=0:\]

\[ \theta(0) = C_1 e^{\lambda_1} + C_2 e^{\lambda_2} = C_1 + C_2 = \theta_b \]
Tip

\[ \frac{\beta C}{h(C_1e^{mx} + C_2e^{-mx})} = -k \frac{d[C_1e^{mx} + C_2e^{-mx}]}{dx} \bigg|_{x=L} \]

\[ = -k \left[ C_1me^{mx} - C_2me^{-mx} \right] \bigg|_{x=L} \]

\[ = -k \left[ C_1me^{ml} - C_2me^{-ml} \right] \]

\[ \Rightarrow h(C_1e^{ml} + C_2e^{-ml}) = -km \left[ C_1me^{ml} - C_2me^{-ml} \right] \]

\[ \theta_b = C_1 + C_2 \]

Two eqns, two unknowns.
Solving simult for \( C_1 + C_2 \)
and sub back into genl soln
we get:

\[ \theta_b = \frac{\cosh m(L-x) + (h/mk) \sinh m(L-x)}{\cosh mL + (h/mk) \sinh mL} \]

This is the case A fin temp.

To get the fin heat transfer rate, we will obtain via Fourier's law at the fin base.
\[ q_f = q_b = -kA_e \left. \frac{dT}{dx} \right|_{x=0} - kA_e \left. \frac{dT}{dx} \right|_{x=L} \]

Using our temp diff for case A, (above) we get \( q_f \)

\[ q_f = \sqrt{hPka_e} \Theta_b \frac{\sinh mL + (h/mk) \cosh mL}{\cosh mL + (h/mk) \sinh mL} \]

This also equals to the heat lost from the fin surf. by convection

\[ = \int_{A_f} h(\Theta(x)) \, dA \]

**CASE B:** Second tip condition is that of insulated fin \( \Theta_f = \text{adiabatic} \) end.

\[ \Theta_f = \frac{\cosh mL (L-x)}{\cosh mL} \]

Similar procedure yields:

\[ \Theta = \frac{\cosh mL (L-x)}{\cosh mL} \]

and:

\[ q_f = \sqrt{hPka_e \Theta_b \tan mL} \]
Similarly, we obtain for case C + D:

\[ \Theta = \left( \frac{\Theta_1}{\Theta_0} \right) \sinh mx + \sinh m(\ell - x) \]

\[ \Theta_0 \]

\[ q_f = \sqrt{hPKAC \Theta_0} \cdot \frac{\cos mh \ell - \Theta_1/\Theta_0}{\sinh mh} \]

and for case D: (Very long fin)

As \( \ell \to \infty \), \( \Theta_0 \to 0 \) (eg, \( T(c) - T_{\infty} = 0 \) or Tip temp \( = T_0 \))

\[ \Theta_0 = e^{-mx} \]

\[ \Theta_0 \]

and:

\[ q_f = \sqrt{hPKAC \Theta_0} \]
<table>
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<tr>
<th>Case</th>
<th>Tip Condition $(x = L)$</th>
<th>Temperature Distribution $\theta/\theta_e$</th>
<th>Fin Heat Transfer Rate $q_f$</th>
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<tbody>
<tr>
<td>A</td>
<td>Convection heat transfer: $h\theta(L) = -k d\theta/dx\big</td>
<td>_{x=L}$</td>
<td>$\cosh mL - (h/\alpha k) \sinh mL - x$ $\frac{\cosh mL}{\cosh mL + (h/\alpha k) \sinh mL}$</td>
</tr>
<tr>
<td>B</td>
<td>Adiabatic $d\theta/dx\big</td>
<td>_{x=L} = 0$</td>
<td>$\cosh mL - x$ $\frac{\cosh mL}{\cosh mL}$</td>
</tr>
<tr>
<td>C</td>
<td>Prescribed temperature: $\theta(L) = \theta_e$</td>
<td>$(\theta/\theta_e) \sinh mL + \sinh mL - x$ $\frac{\sinh mL}{\sinh mL}$</td>
<td>$M \frac{\cosh mL - \theta/\theta_e}{\sinh mL}$ $\frac{M}{\sinh mL}$ $(3.77)$ $(3.78)$</td>
</tr>
<tr>
<td>D</td>
<td>Infinite fin $(L \to \infty)$: $\theta(L) = 0$</td>
<td>$e^{-\alpha x}$ $\frac{e^{-\alpha x}}{\alpha}$</td>
<td>$M$ $\frac{M}{\alpha}$ $(3.79)$ $(3.80)$</td>
</tr>
</tbody>
</table>

\[\begin{align*}
\theta &= T - T_w \\
\theta_e &= \theta(0) = T_b - T_w \\
m^2 &= hP/\alpha k \\
M &= \sqrt{kP/\alpha \theta_b}
\end{align*}\]